Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 6, 1999

NUMERICAL ANALYSIS OF THE FLUTTER OF A SHALLOW SHELL

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UDC 539.3:534.1

The stability of vibrations of a shallow shell that is rectangular in plan in a gas flow is studied for an arbitrary vector of flow velocity. Mathematically, the problem is shown to reduce to an ill-conditioned computational problem. To solve this problem, we propose a numerical saturation-free algorithm that allows one to obtain a sufficiently accurate solution for a grid containing 169 (13 × 13) *nodes. In calculations for cylindrical and spherical shallow shells, new mechanical effects concerning the vibration modes and the dependence of the critical flutter velocity on the direction of the flow-velocity vector were found.*

Introduction. Il'yushin and Kiyko [1] gave a new formulation of the problem of the flutter of a shallow shell under the assumption that the differential pressure acting on the shell is determined within the framework of the law of plane sections in supersonic aerodynamics [2]. The mathematical problem reduces to an eigenvalue problem for a system of two equations with biharmonic higher operators with respect to the amplitude deflection φ and stress F functions. For certain boundary conditions, the function F can be eliminated (numerically); the other equation for φ contains two dimensionless constants of orders 10^{-3} and 10^2 (for the characteristic values of the parameters) for higher-order derivatives, which predetermines the fact that the problem is ill-conditioned. At the same time, the presence of the boundary layer in the solution indicates the need to refine the grid in the neighborhood of the contour.

The saturation-free method proposed by Babenko is ideal to overcome the above-mentioned computational difficulties; it has been applied successfully in the analysis of the flutter of a plate with an arbitrary plan form [3]. In the present paper, the method is generalized to the problem of a panel flutter of rectangular-in-plan shallow shells; calculations were performed for cylindrical and spherical shells.

1. Formulation of the Problem. The initial system of differential equations can be written in dimensionless form [1]

$$D\Delta^2\varphi - hL(F) - k(\boldsymbol{v}, \operatorname{grad}\varphi) = \lambda\varphi, \qquad \Delta^2 F + EL(\varphi) = 0; \tag{1.1}$$

$$\lambda = -\rho h\omega^2 - k\omega; \tag{1.2}$$

$$L(f) = k_y \frac{\partial^2 f}{\partial x^2} + k_x \frac{\partial^2 f}{\partial y^2}, \tag{1.3}$$

where $D = Eh^3/(12(1-\nu^2))$ is the flexural rigidity, E is Young's modulus, ν is Poisson's ratio, k is the polytropic index, $\boldsymbol{v}$ is the velocity of the air flow, h is the thickness of the shell, ρ is the density of the shell material, ω is the complex frequency of vibrations, k_x and k_y are the principal curvatures (the lines of principal curvatures coincide with the coordinate lines), $\varphi = \varphi(x,y)$ is the deflection of the shell, and $F = F(x,y)$ is a stress function. All the above-mentioned quantities are dimensionless. Nondimensionalization was performed in the same manner as in [3].

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 0021-8944/99/4006-1082 $22.00

The above equations are considered in the region $G = \{-1 \leqslant x \leqslant 1,\ -b \leqslant y \leqslant b\}$, i.e., in a rectangle; the lines of the shell's principal curvatures coincide with the coordinate lines.

We solve Eqs. (1.1) and (1.2) for two types of boundary conditions.

(1) Hinged (simply supported) edges:

$$\varphi = 0, \quad \frac{\partial^2 \varphi}{\partial x^2} = 0, \quad \frac{\partial^2 F}{\partial y^2} = 0, \quad \frac{\partial^2 F}{\partial x\, \partial y} = 0 \quad \text{for} \quad x = 1,\ -1,$$

$$\varphi = 0, \quad \frac{\partial^2 \varphi}{\partial y^2} = 0, \quad \frac{\partial^2 F}{\partial x^2} = 0, \quad \frac{\partial^2 F}{\partial x\, \partial y} = 0 \quad \text{for} \quad y = b,\ -b;$$

(2) Clamped slipping edges:

$$\varphi = 0, \quad \frac{\partial \varphi}{\partial x} = 0, \quad \frac{\partial^2 F}{\partial y^2} = 0, \quad \frac{\partial^2 F}{\partial x\, \partial y} = 0 \quad \text{for} \quad x = 1,\ -1,$$

$$\varphi = 0, \quad \frac{\partial \varphi}{\partial y} = 0, \quad \frac{\partial^2 F}{\partial x^2} = 0, \quad \frac{\partial^2 F}{\partial x\, \partial y} = 0 \quad \text{for} \quad y = b,\ -b.$$

It can easily be shown that the boundary conditions imposed on the stress function F can be replaced, without loss of generality, by the equivalent conditions [4]

$$(x, y) \in \partial G, \qquad F = 0, \qquad \frac{\partial F}{\partial n} = 0, \tag{1.4}$$

where n is the outward normal vector to the contour of the shell.

The vibrations of the shell are stable or unstable, depending on whether $\mathrm{Re}\,\omega < 0$ or $\mathrm{Re}\,\omega > 0$. If $\lambda = \alpha + i\beta$ is the eigenvalue of the formulated problem, in view of (1.2) the inequalities imply that $f(\alpha, \beta) > 0$ or $f(\alpha, \beta) < 0$, where $f(\alpha, \beta) = \alpha k^2 - \rho h \beta^2$. Inasmuch as $\alpha = \alpha(v_x, v_y)$ and $\beta = \beta(v_x, v_y)$, where $v_x = v \cos\theta$, $v_y = v \sin\theta$, and $v = |\boldsymbol{v}|$, the equation $f(\alpha, \beta) = 0$ determines the neutral curve (stability parabola) in the complex plane λ and the corresponding critical flutter velocity v for a specified θ.

If $v = 0$, all the eigenvalues are real; as the flow velocity increases, some eigenvalues enter the complex plane. Consequently, the problem is to find (for a given θ) the first complex eigenvalue that crosses the stability parabola. As a result, the critical velocity and the corresponding vibration mode (eigenfunction) are determined. It follows that to solve the problem correctly, it is necessary to determine a sufficiently long initial part of the spectrum.

Thus, to determine the root of the equation $f(\alpha, \beta) = 0$, in each iteration one has to solve the complete eigenvalue problem for an $N \times N$ nonsymmetric matrix, where N is the number of grid nodes. The difficulties are overcome by the saturation-free method, which provides good accuracy for smooth solutions even when a relatively coarse grid is used. The eigenvalues of the matrix were calculated by the QR-algorithm (the program implementation in the EISPACK package).

2. Discretization. Discretization of the boundary-value problems described is associated with discretization of the biharmonic operators $\Delta^2\varphi$ and $\Delta^2 F$ under the boundary conditions of simple support (clamping) and the boundary condition of clamping, respectively. Moreover, the operator $L(f)$ and the terms containing first-order derivatives $k(\boldsymbol{v}, \operatorname{grad}\varphi)$ should be discretized.

We assume that k_x and k_y are constants. We have $k_x = 0$ and $k_y = 1/R$ for a cylindrical shell and $k_x = k_y = 1/R$ for a spherical shell (R is the radius of the shell). Therefore, discretization of the operator $L(f)$ under the Dirichlet homogeneous boundary condition is needed. This discretization is performed according to [5]. The terms with the first-order derivatives $k(\boldsymbol{v}, \operatorname{grad}\varphi)$ are also discretized according to [5]. It should be noted that since $L(f)$ is the second-order operator, it suffices to satisfy the only boundary condition, namely, $f = 0$ on ∂G to discretize this operator. Since there are no nodes at the boundary, an interpolation formula that does not satisfy (compulsorily) the boundary condition $\varphi = 0$ on ∂G was used for discretizing the terms with first-order derivatives. In discretization of the biharmonic operators, both boundary conditions were satisfied. Calculations show that the solution obtained for φ satisfies the boundary conditions.

Discretization of the biharmonic operator under the boundary conditions of simple support is described in [5]. We consider discretization of the biharmonic operator under the boundary condition of clamping (1.4). For the function $F = F(x, y)$ in a rectangle, we use the interpolation formula

$$
\begin{gathered}
F(x,y) = \sum_{j=1}^{n}\sum_{i=1}^{m} M_{i0}(z)L_{j0}(x)F(x_j, y_i), \qquad y = bz, \qquad z \in [-1,1], \qquad x \in [-1,1], \\
L_{j0}(x) = \frac{l(x)}{l'(x_j)(x - x_j)}, \qquad l(x) = (x^2-1)^2 T_n(x), \qquad x_j = \cos\theta_j, \\
\theta_j = \frac{(2j-1)\pi}{2n}, \quad j = 1,2,\dots,n, \qquad M_{i0}(z) = \frac{M(z)}{M'(z_i)(z - z_i)}, \\
M(z) = (z^2-1)^2 T_m(z), \qquad z_i = \cos\theta_i, \qquad \theta_i = \frac{(2i-1)\pi}{2m}, \qquad i = 1,2,\dots,m,
\end{gathered}
\tag{2.1}
$$

where T_n is the Chebyshev polynomial. The interpolation formula (2.1) satisfies the boundary conditions (1.4). We enumerate the nodes in the rectangle (x_j, y_i) first along y and then along x, i.e., from top to bottom and from right to left, and substitute (2.1) into (1.1). As a result, we obtain

$$H\varphi - hL^hF = \lambda\varphi; \tag{2.2}$$

$$H_3F + EL^h\varphi = 0. \tag{2.3}$$

Here φ is the vector containing approximate values of the deflection of the shell at the grid nodes and H is an $N \times N$ matrix ($N = mn$ is a matrix of the discrete problem of a plate). Its construction for the case of simple support is described in [5]. For the case of clamping, the construction of the matrix H is similar to that considered above. Expressing F from (2.3) and substituting it into (2.2), we obtain

$$(H + hEL^hH_3^{-1}L^h)\varphi = \lambda\varphi, \tag{2.4}$$

where λ is the approximate eigenvalue and L^h and H_3 are $N \times N$ matrices that appear after discretization of the operators L [see (1.3)] and $\Delta^2 F$.

Further analysis was carried out using the finite-dimensional eigenvalue problem (2.4). As was mentioned in the Introduction, this problem contains a large parameter hE (of order 10^2 for the data used in calculations). The matrix of this parameter is nonsymmetric and can, therefore, have complex eigenvalues for the flow velocity $v = 0$, which is supported by particular calculations. Therefore, the applied approach was updated. The matrix H_3 in (2.4) was replaced by the matrix $H_3 = 0.5(H_3 + H_3')$, where the prime denotes a transposed matrix. This operation can be interpreted as follows. The initial problem is self-adjoint [biharmonic equation with boundary conditions (1.4)]. But, as a result of discretization, we obtain the nonsymmetric matrix H_3. We represent H_3 in the form

$$H_3 = 0.5(H_3 + H_3') + 0.5(H_3 - H_3')$$

and relate the antisymmetric part to the discretization error. The corresponding perturbation introduced into the eigenvalues of the matrix H_3 depends on how the resolvents of the matrices H_3 and $(H_3 + H_3')/2$ are close in the part of the complex plane that is of interest in study of flow stability. This perturbation can be estimated theoretically according to the scheme of [6], but we verified it numerically.

The matrix H_3 of dimension 361×361 ($361 = 19 \times 19$) has the first eigenvalue $\lambda_1 = 604.0638$, which was compared with the result of [7]: $\sqrt{\lambda_1}/\pi^2 = 2.4902$ (2.489). The bracketed value was calculated in [7]. The matrix $(H_3 + H_3')/2$ has the eigenvalue $\lambda_1 = 559.242064$, where $\sqrt{\lambda_1}/\pi^2 = 2.3961$.

Thus, the perturbation introduced into the eigenvalues by matrix symmetrization is acceptable. A similar symmetrization was applied to the matrices L^h and H_0 (H_0 is a matrix of the discrete biharmonic operator for the deflection φ). As a result, the matrix $L^hH_3^{-1}L^h$ became symmetric with an accuracy of 10^{-6}.

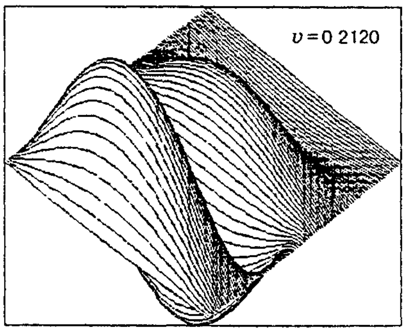

Fig. 1

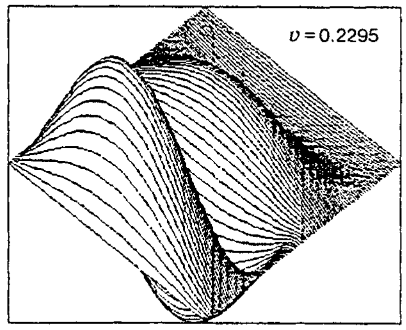

Fig. 2

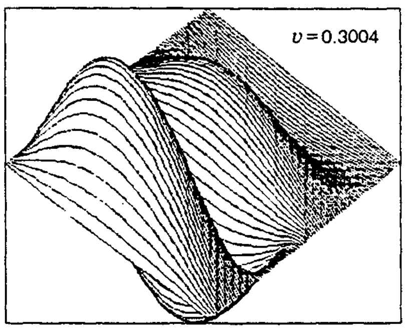

Fig. 3

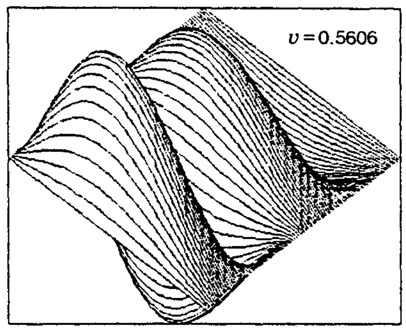

Fig. 4

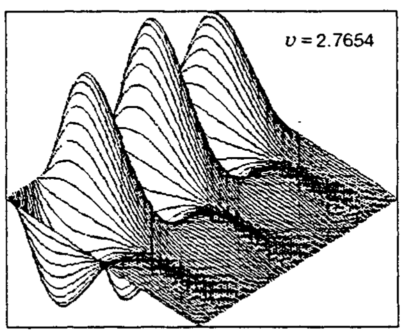

Fig. 5

This, however, was insufficient, since the discrete problem still had complex eigenvalues for $v = 0$. After the repeated symmetrization of the matrix $L^h H_3^{-1} L^h$, the eigenvalues of the discrete problem for $v = 0$ became real and positive. In calculating the critical velocity, convergence was observed.

3. Numerical Results. Calculations were performed at the following values of the parameters: $k = 1.4$, $\nu = 0.33$, $c_0 = 331.26$ m/sec, $p = 1.0133 \cdot 10^5$ Pa, $E = 6.867 \cdot 10^{10}$ Pa, and $\rho = 2.7 \cdot 10^3$ kg/m^3. The relative thickness and dimensionless radius of the shell were taken to be $3 \cdot 10^{-3}$ and 2.5, respectively. Preliminary calculations using 9×9, 13×13, and 19×19 grids showed that the results for the 13×13 and 19×19 grids are close. Below, we give the critical velocities obtained for a 19×19 grid.

Calculations for a square-in-plan spherical shallow shell can be regarded as testing calculations. For the angles $\theta = 0$, $\pi/8$, $\pi/4$, $3\pi/8$, and $\pi/2$, the following critical velocities were obtained: 1.4263 (20), 1.4924 (18), 1.5813 (18), 1.4924 (18), and 1.4263 (20). The bracketed numbers indicate the first eigenvalue that enters the stability parabola. As was expected from the symmetry of the problem, the critical velocities are symmetric about the straight line $\theta = \pi/4$. This supports the correctness of the method and the program. Furthermore, to control the calculations, the following two diagrams were plotted: the forms of the deflection function $\operatorname{Re}\varphi(x,0)$ and $\operatorname{Re}\varphi(0,y)$ and the form of the two-dimensional function $\operatorname{Re}\varphi(x,y)$. The curves $\operatorname{Re}\varphi(x,0)$ and $\operatorname{Re}\varphi(0,y)$ coincided for $\theta = \pi/4$; the eigen-forms $\operatorname{Re}\varphi(x,y)$ for the angles $\theta = 0$ and $\theta = \pi/2$ were identical as well. This shows that the calculations are correct.

For a clamped spherical shallow shell, for the same directions of the flow-velocity vector the following critical velocities were obtained: 1.6424 (20), 1.7038 (16), 1.6876 (17), 1.7038 (16), and 1.62384 (20). Generally, the results are similar to those obtained in the previous case.

Calculations for a rectangular-in-plan ($b = 0.5$) spherical shallow shell were performed. For the case of simple support, the following critical velocities were obtained for the same values of the angle θ: 1.7752 (9), 1.8787 (9), 1.8414 (5), 1.8558 (4), and 1.7469 (4). For the boundary conditions of clamping, we obtained 1.6138 (9), 1.6902 (9), 1.8935 (5), 1.7335 (5), and 1.6602 (5), respectively.

Further calculations were performed for a square-in-plan cylindrical, simply supported shell. For the same values of the angle θ, we obtained the following critical velocities: 2.7654 (7), 0.5606 (1), 0.3004 (1), 0.2295 (1), and 0.2120 (1). The principal difference is seen between these and the preceding results: the velocity decreases abruptly for θ close to $\pi/2$. We note that, for a square plate, the critical velocity is equal to 0.2103 at $\theta = 0$ and $\pi/2$ (the calculations were performed by the above-described method for $k_y = 0$). Thus, the critical flutter velocity of the flow along the generatrix of the cylindrical shell is one order of magnitude greater than that across the generatrix. This effect can easily be explained: the bending rigidity of a cylindrical shell along the generatrix is much greater than that across the generatrix. The evolution of the eigen-forms is shown in Figs. 1–5. In addition, cylindrical shells with radii 10 and 40 were calculated for the same angles of direction of the flow-velocity vector. The values of the critical velocity obtained for a radius equal to 10 are 0.8216 (14), 0.4629 (1), 0.2287 (1), 0.1727 (1), and 0.1591 (1) and those obtained for radius 40 are 0.3378 (6), 0.3439 (1), 0.2433 (1), 0.1673 (1), and 0.1514 (1). Thus, as $R \to \infty$, the critical flutter velocity decreases in flows directed both along and across the generatrix. This conclusion is very important, since the small initial convexity of the shell in a transverse flow (for a radius equal to 40, the shell rise is 0.0125) decreases the critical flutter velocity.

4. Conclusions. An experimental algorithm for the complex computational problem of calculation of the critical flutter velocity of the shallow shell has been described. The calculations show good accuracy for a grid containing $169 = 13 \times 13$ nodes. The results were obtained by Babenko's saturation-free discretization method. All the results concerning the mechanics are new. Ogibalov and Koltunov [8] investigated the flutter of a spherical shallow shell by the Bubnov–Galerkin method. As is well known, this method gives underestimated values of the critical velocity. A new mechanical effect for a cylindrical panel has been found, namely, the abrupt change in the critical velocity as the angle θ is varied.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 97-01-00923).

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